1. The Dual Lambda Algebra

We will use Singer’s dual lambda algebra introduced in [6]. Let $GL_s = GL_s(F_2)$ act on $F_2[t_1, \ldots, t_s]$ in the natural way. We have the subgroup $B_s \subset GL_s$ of upper triangular matrices. Then by [5] the $B_s$-invariants in $F_2[t_1, \ldots, t_s]$ in a polynomial algebra with generators

$$V_k = \prod(a_1 t_1 + \cdots + a_{k-1} t_{k-1} + t_k)$$

for $k = 1, \ldots, s$, where the product is over all choices $a_i \in F_2$.

Let $D_s = V_1 V_2 \cdots V_s$ be the Dickson invariant, which is an invariant of $GL_s$. Then $GL_s$ also acts on $F_2[t_1, \ldots, t_s, D_s]$. We define $\Delta_s$ to be its $B_s$-invariants and $\Gamma_s$ to be its $GL_s$-invariants. One finds $\Delta_s = F_s[V_1^\pm, \ldots, V_s^\pm]$.

Define another set of generators of $\Delta_s$ to be $v^i_k = V^i_k V_1 \cdots V_{k-1}$. Let $\Delta^+_s$ to be the span of the elements of the form $v^i_1 \cdots v^i_s$ in which at least one of the $i_j < 0$. Let $\Gamma^+_s = \Gamma_s \cap \Delta^+_s$, and $\Gamma^+_s = \Gamma_s / \Gamma^+_s$.

Define $\partial : \Delta_s \to \Delta_{s-1}$ to be the map $Res_{v_s}$, which takes the coefficient before $v_s^{-1}$. It is proved in [6] that the restriction of $\partial$ to $\Gamma$ makes $\Gamma$ into a complex, and $\Gamma^+$ is a direct summand of $\Gamma$ as a complex.

There is a coproduct $\psi$ on $\Delta$ defined by $\psi_{p,q} : \Delta_{p+q} \to \Delta_p \otimes \Delta_q$ where $\psi_{p,q}$ is an algebra isomorphism sending $v_i$ to $v_i \otimes 1$ for $i \leq p$ and to $1 \otimes v_{i-p}$ for $i > p$. It is proved in [6] that $\Gamma$ is a subcoalgebra of $\Delta$ and $\Gamma^+$ is a quotient coalgebra of $\Gamma$.

It is proved in [6] that $\Gamma^+$ is dual to the lambda algebra, with $v^i_1 \cdots v^i_s$ dual to $\lambda_i \ldots \lambda_1$.

2. The Hecke Algebra

We recall the basics of the mod 2 Hecke algebra in this section, following [2] and [3].

Let $H_s$ be the mod 2 Hecke algebra. It has a set of basis $\{T_w\}$ for $w$ running through the Weyl group $W_s$. $H_s$ acts on the $B_s$-invariants in any $GL_s$-module such that $T_w$ acts as $B_s w \in F_2[GL_s]$. As an algebra, $H_s$ is generated by $T_{w_i}$ for $i = 1, \ldots, s-1$, where $w_i$ is the transposition of the $i^{th}$ and $(i+1)^{st}$ element in the canonical basis of $F_2^s$. We also define $e_s = T_{w_0}$ where $w_0$ inverts the order of the elements in the canonical basis. Let $\hat{e}_s = \sum w \in W_s T_w$. Both $e_s$ and $\hat{e}_s$ are idempotents mod 2. For any $GL_s$-module $M$, we can identify its $GL_s$ invariants with $\hat{e}_s M^{B_s}$. We can also identify (or define) its Steinberg sumand to be $e_s M^{B_s}$. 

Proposition 1. The generators $T_w$ satisfy the following relations mod 2, and these relations define $\mathcal{H}$:

(1) $T_w^2 = T_w$.
(2) $T_{w_i}T_{w_{i+1}}T_{w_i} = T_{w_{i+1}}T_{w_i}T_{w_{i+1}}$.
(3) $T_{w_i}T_{w_j} = T_{w_j}T_{w_i}$ for $|i - j| \geq 2$.

From this proposition, we can define a map $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}_{s+t}$ by sending $T_{w_i} \otimes 1$ to $T_{w_i}$, and $1 \otimes T_{w_i}$ to $T_{w_{i+1}}$. Define $e_t, \hat{e}_s \in \mathcal{H}_{s+t}$ to be the image of $1 \otimes e_t$ and $\hat{e}_s \otimes 1$ of this map respectively.

More generally, define $e_{i,j}$ to be the image of $1 \otimes e_{j+1-i} \otimes 1$ under the map $\mathcal{H}_{j-i+1} \otimes \mathcal{H}_{s-i-j} \to \mathcal{H}_s$.

Proposition 2. The elements $e_i$ and $\hat{e}_i$ in $\mathcal{H}_{s+t}$ satisfy the following relations:

(1) $e_t \hat{e}_s = \hat{e}_s e_t$.
(2) $e_t e_i = e_t e_{i+1}$ for $i \leq t$.
(3) $\hat{e}_i \hat{e}_s = \hat{e}_s \hat{e}_i$ for $i \leq s$.
(4) $\hat{e}_s e_{i+1} e_i + e_i e_{s+1} e_i = \hat{e}_s e_t$.

Define $\hat{T}_{w_i} = 1 + T_{w_i}$.

Proposition 3. In $\mathcal{H}_s$, we have

(1) $\hat{T}_{w_i} \hat{e}_s = \hat{e}_s \hat{T}_{w_i}$.
(2) $\hat{e}_2 = \hat{T}_{w_1}$.
(3) $\hat{e}_k = \hat{T}_{w_1} \ldots \hat{T}_{w_{k-1}} \hat{e}_{k-1}$.

In the following we will abbreviate $T_{w_i}$ by $T_i$, etc.

Now specialize to the case $\Delta_s$. $\mathcal{H}_s$ acts on $\Delta_s$, and one can identify $\Gamma_s$ as $\hat{e}_s \Delta_s$. The map $\psi_{p,q}$ preserves the action of $\mathcal{H}_p \otimes \mathcal{H}_q$. The map $\partial_s$ preserves the action of $\mathcal{H}_{s-1}$.

3. The complexes $\mathcal{L}(n)$

In this section, we define complexes $\mathcal{L}(n)$ which are isomorphic to the dual of $\Lambda$-modules of the spectra $L(n)$.

Define $\mathcal{L}(n)_s = \hat{e}_{s-n} e_n \Delta_s$ for $s \geq n$ and $\mathcal{L}(n)_s = 0$ for $s < n$. Define the differential by the formula $\partial_s : \mathcal{L}(n)_s \to \mathcal{L}(n)_{s-1}$ to be $\partial_s(x) = Res_{v_s}(T_{s-1} \ldots T_{s-n-1} x)$ for any $x \in \mathcal{L}(n)_s \subset \Delta_s$.

Proposition 4. The map $\partial'$ lands in $\mathcal{L}(n)$.

Proof. It is trivial to check $\hat{e}_{s-n-1} T_{s-1} \ldots T_{s-n} x = \hat{T}_{s-1} \ldots \hat{T}_{s-n} x$. To complete the proof, it remains to check the equations $\hat{T}_k \hat{T}_{s-1} \ldots \hat{T}_{s-n} x = 0$ for $k = s - n, \ldots, s - 2$. This can be done with the commutation relations of the $T_i$’s. First we can transform the expression by moving $T_k$ rightward to arrive at the expression $T_{s-1} \ldots T_{k+1} T_k \ldots T_{s-n} x$. Using the braid relation, this equals $T_{s-1} \ldots T_{k+1} T_{k-1} \ldots T_{s-n} x$. Then we move the right $\hat{T}_{k+1}$ further rightward, and end with $T_{s-1} \ldots T_{k+1} \hat{T}_{k+1} \ldots T_{s-n} T_{k+1} x$. Finally observe $\hat{T}_{k+1} x = 0$ for $k = s - n, \ldots, s - 1$.

Define $\mathcal{L}(n)^+_s = \mathcal{L}(n)_s / \mathcal{L}(n)_{s+1} \cap \Delta^-$. 
**Theorem 1.** The map \( \partial' \) satisfies \( \partial'^2 = 0 \). Moreover, \( \mathcal{L}(n)^+ \) is isomorphic to the dual of \( \Lambda \)-modules of the spectra \( L(n) \).

**Proof.** The first assertion is a consequence of Proposition 3.1 in [6], which implies \( \partial'^2 \) restricted to the image of \( \hat{T}_{s-1} \) is zero, provided we can show any expression \( \hat{T}_{s-2} \cdots \hat{T}_{s-n-1} \hat{T}_{s-1} \cdots \hat{T}_{s-n}x \) can be transformed into an expression starting with \( \hat{T}_{s-1} \). To do this, first move \( \hat{T}_{s-n-1} \) rightward to arrive at the expression \( \hat{T}_{s-2} \cdots \hat{T}_{s-n} \hat{T}_{s-1} \cdots \hat{T}_{s-n-1} \hat{T}_{s-n}x \). We know \( x = \hat{T}_{s-n-1}x \). So \( \hat{T}_{s-n-1} \hat{T}_{s-n}x = \hat{T}_{s-n} \hat{T}_{s-n-1} \hat{T}_{s-n}x \). So we get an expression starting with \( \hat{T}_{s-2} \cdots \hat{T}_{s-n} \hat{T}_{s-1} \cdots \hat{T}_{s-n+1} \hat{T}_{s-n} \). Then an induction completes the proof.

The second assertion can be proved by comparing with the formula of the action of the Steenrod algebra on \( H^*(L(n)) \) described in [1] using the Nishida relations. \( \square \)

4. **The Kuhn-Priddy map and the transfer map**

We define the Kuhn-Priddy map \( s_n : \mathcal{L}(n+1) \to \mathcal{L}(n) \) by the formula \( s_n(x) = \hat{e}_{s-n}x \) for \( x \in \mathcal{L}(n+1)_s \).

**Proposition 5.** The map \( s_n \) is a map of complexes.

**Proof.** Let \( x \in \mathcal{L}(n+1)_s \). Then

\[
\partial'(s_n(x)) = Res_{v_s} \hat{T}_{s-1} \cdots \hat{T}_{s-n} \hat{e}_{s-n} x
\]

We have \( x = \hat{e}_{s-n}x \), and \( \hat{e}_{s-n} = \hat{e}_{s-n-1} \hat{T}_{s-n-1} \hat{e}_{s-n-1} \), so

\[
\hat{T}_{w_{s-1}} \cdots \hat{T}_{w_{s-n}} \hat{e}_{s-n} x = \hat{e}_{s-n} \hat{T}_{s-1} \cdots \hat{T}_{s-n-1} x
\]

because \( \hat{e}_{s-n-1} \) commutes with \( \hat{T}_{s-n}, \ldots, \hat{T}_{s-1} \).

On the other hand, we have

\[
s_n(\partial' x) = Res_{v_s} \hat{e}_{s-n} \hat{T}_{s-1} \cdots \hat{T}_{s-n-1} x \]

\( \square \)

Similarly, we define the transfer map \( d_n : \mathcal{L}(n) \to \mathcal{L}(n+1) \) by the formula \( d_n(x) = e_{n+1}x \) for \( x \in \mathcal{L}(n+1)_s \).

**Proposition 6.** The map \( d_n \) is a map of complexes.

**Proof.** Let \( x \in \mathcal{L}(n)_s \). Then

\[
\partial'(d_n(x)) = Res_{v_s} \hat{T}_{s-1} \cdots \hat{T}_{s-n-1} e_{n+1} x
\]

We have \( e_{n+1} = e_n + e_n \hat{T}_{s-n} e_n \), and \( e_n x = x = \hat{T}_{s-n-1} x \). So

\[
\hat{T}_{s-1} \cdots \hat{T}_{s-n-1} e_{n+1} x = \hat{T}_{s-1} \cdots \hat{T}_{s-n} x + \hat{T}_{s-1} \cdots \hat{T}_{s-n-1} e_n \hat{T}_{s-n} x
\]

Because \( e_n \) commutes with \( \hat{T}_{s-n-1} \), we have

\[
\hat{T}_{s-1} \cdots \hat{T}_{s-n-1} e_n \hat{T}_{s-n} x = \hat{T}_{s-1} \cdots \hat{T}_{s-n} e_n \hat{T}_{s-n-1} \hat{T}_{s-n} x
\]

On the other hand, we have

\[
d_n(\partial' x) = Res_{v_s} e_{s-n-1} \hat{T}_{s-1} \cdots \hat{T}_{s-n} x
\]

We have

\[
e_{s-n-1,s-1} = e_{s-n,s-1} + e_{s-n,s-1} \hat{T}_{s-n-1} e_{s-n,s-1}
\]
From the proof of Theorem 1, we know
\[ e_{s-n,s-1} \hat{T}_{s-1} \ldots \hat{T}_{s-n-1} x = \hat{T}_{s-1} \ldots \hat{T}_{s-n-1} x \]
So
\[ e_{s-n-1,s-1} \hat{T}_{s-1} \ldots \hat{T}_{s-n} x = \hat{T}_{s-1} \ldots \hat{T}_{s-n} x + e_{s-n,s-1} \hat{T}_{s-n-1} \hat{T}_{s-1} \ldots \hat{T}_{s-n-1} x \]
We also have
\[ e_{s-n,s-1} \hat{T}_{s-n-1} \hat{T}_{s-1} \ldots \hat{T}_{s-n} x = e_{s-n,s-1} \hat{T}_{s-1} \ldots \hat{T}_{s-n-1} \hat{T}_{s-n} x \]
Then an induction on \( n \) proves the proposition once we notice
\[ e_{n-1} \hat{T}_{s-n-1} \hat{T}_{s-n} x = \hat{T}_{s-n-1} \hat{T}_{s-n} x \]
□

The following is a direct consequence of the formula
\[ \hat{e}_{s} e_{t+1} \hat{e}_{s} + e_{t} \hat{e}_{s+1} e_{t} = \hat{e}_{s} e_{t} \]

**Proposition 7.** We have \( d_{n} s_{n} + s_{n-1} d_{n-1} = 1 \).

This is consistent with the algebraic Whitehead conjecture proved in [4].

**References**